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REGULARITY PROPERTIES OF VISCOSITY SOLUTIONS OF A NON-HÖRMANDER DEGENERATE EQUATION ☆

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ABSTRACT. – We study the interior regularity properties of the solutions of a nonlinear degenerate equation arising in mathematical finance. We set the problem in the framework of Hörmander type operators without assuming any hypothesis on the degeneracy of the associated Lie algebra. We prove that the viscosity solutions are indeed classical solutions. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous étudions la régularité intérieure des solutions de viscosité d'une équation non linéaire du second ordre dégénérée que l'on rencontre en finance mathématique. Nous étudions le problème par la théorie des opérateurs de Hörmander sans aucune hypothèse sur la dégénerescence de l'algèbre de Lie engendrée. Nous montrons que la solution de viscosité est une solution classique. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

In this paper we prove some regularity results for viscosity solutions of a nonlinear differential equation arising in mathematical finance. In [1], Antonelli, Barucci and Mancino introduce a new model for agent's decision under risk, in which the utility function u is a solution of the Cauchy problem

$$Lu = f, \quad \text{in } S_T \equiv \mathbb{R}^2 \times]0, T[, \tag{1.1}$$

$$u(\cdot, 0) = g, \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

where T is suitably small and L is the nonlinear operator defined by

$$Lu = \partial_{xx} u + u \partial_y u - \partial_t u,$$

and $(x, y, t) = z$ denotes the point in $\mathbb{R}^3$.

In the same paper the authors prove by means of probability methods the existence of a continuous viscosity solution, in the sense of the User's guide [13], of (1.1)–(1.2), satisfying

$$\bigl|u(x, y, t) - u(\xi, \eta, \tau)\bigr| \leqslant C_T \bigl(|x - \xi| + |y - \eta|\bigr) \tag{1.3}$$

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for every $(x, y), (\xi, \eta) \in \mathbb{R}^2$, $t \in [0, T[$, under the assumption that f and g are uniformly Lipschitz continuous functions. On the other hand, in a recent paper [2], Antonelli and Pascucci prove that the solution u found in [1] can be also considered as a distributional solution.

Other existence results for weak solutions of the Cauchy problem for a more general class of equations, that contains (1.1), are obtained in [31] and [14]. This kind of solutions, however, is few regular and does not satisfy condition (1.3).

Related problems arise in the stochastic control theory. For instance, the value function v of a suitable control problem is a semiconcave solution of the following Cauchy problem:

$$\partial_{xx} v + \frac{1}{2}(\partial_y u)^2 - \partial_t v = \varphi, \quad \text{in } S_T \equiv \mathbb{R}^2 \times]0, T[,$$
$$v(\cdot, 0) = \psi, \quad \text{in } \mathbb{R}^2,$$

for some continuous functions φ and ψ (see [16]). Note that the function $u = \partial_y v$ is, at least formally, a solution of our Cauchy problem (1.1)–(1.2) and the regularity of v is presently object of study in [7].

Here are we are interested in the interior regularity of the solution u found in [1] and [2]. Since L is a degenerate second-order operator, regularity results proved in [6,32,33] for viscosity solutions, and in [24] for weak solutions, do not apply. Instead we will study the regularity in the framework of Hörmander type operators, representing the operator L as a sum of squares of vector fields plus a first-order term:

$$L = \partial_x^2 + Y, \quad Y = u\partial_y - \partial_t. \tag{1.4}$$

General operators of this kind can be represented as follows:

$$\sum_{i,j} a_{ij} X_i X_j u + X_0 u = f, \tag{1.5}$$

where X_i are linear smooth vector fields and $a_{ij} \in C^\infty$ and the rank of the Lie algebra generated by X_j is maximal at every point (see [20]). The main properties of the operator in (1.5) (such as existence of a fundamental solution, control distance) have been established in [27,29,30, 21] (see also [25] for a particular class of operators with the same structure as L). Using these properties, a general theory of the regularity both in Sobolev spaces and in spaces of Hölder continuous functions has been settled down in [17,19] and [29]. See also [15] for related results, for pseudodifferential operators. A Morrey type result is proved in [22]. The regularity of solutions with less regular coefficients has been studied in [5,4] in Sobolev spaces with the technique introduced in [8]. In [28,26] and [23], it is considered the case of spaces of Hölder continuous functions. Then operators of the form (1.5) with nonlinear vector fields X_i have been studied in [9,11]. In all these papers the regularity properties of the solution relie on the fact that the Lie algebra generated by the vector fields has maximum rank at every point.

Concerning operator L in the form (1.4), the commutator of ∂_x and Y is

$$[\partial_x, Y] = u_x \partial_y \tag{1.6}$$

and an Hörmander condition can be expressed as

$$\partial_x u(z) \neq 0, \quad \forall z \in \Omega. \tag{1.7}$$

Indeed, in [12] the authors proved:

THEOREM 1.1. – *Let Ω be an open set in $\mathbb{R}^3$ and u a classical solution of* (1.1) *on Ω with $f \in C^\infty(\Omega)$. If* (1.7) *holds, then $u \in C^\infty(\Omega)$.*

The first results for non-Hörmander linear vector fields X_i are contained in [18]. In [10] a regularity result was established without conditions on the commutators, for a nonlinear equation with a structure different from L. Here we further develop this idea and we prove that the viscosity solutions are classical solutions of equation (1.1). The main interest in our treatment lies in the fact that, in spite of the degeneracy of the operator, we do not require any assumption on the commutators. In particular we do not require any more condition (1.7). Hence the Lie algebra associated to the operator is completely unknown. However we consider L as a subelliptic operator with respect to some tentative Lie groups, and we use a representation formula for the solution u in this setting. This allows us to prove the existence of the directional Euclidean derivative of u

$$Yu(z) = \frac{\partial u}{\partial \nu_z}(z),$$

where $\nu_z = (0, u(z), -1)$.

In Section 2 we prove some preliminary results. Using standard techniques, we get:

PROPOSITION 1.2. – *If u is a viscosity solution of* (1.1) *satisfying* (1.3)*, then it is a strong solution of the same equation, in the sense that*

$$u \in H^1_{\mathrm{loc}}(S_T), \qquad u_{xx} \in L^2_{\mathrm{loc}}(S_T),$$

and the equation is satisfied a.e.

Moreover u_x is a strong solution of the linear equation formally obtained by differentiating (1.1).

In Section 3 we prove our main results. Here we use the deep geometric properties of some Hörmander operators naturally associated to L and we prove that the weak derivatives of u can be computed pointwise. More precisely we have the following:

THEOREM 1.3. – *If u is a strong solution of* (1.1) *satisfying* (1.3) *then it is a classical solution, in the sense that u_{xx}, Yu are continuous and the equation is pointwise satisfied.*

A similar regularity result also holds for u_x.

Finally, in Section 4, using the properties of u_x established in the previous sections and a propagation principle, we prove a sufficient condition for (1.7) to hold. Precisely we prove:

THEOREM 1.4. – *Assume that $f \in C^1 \cap \mathrm{Lip}(S_T)$ is such that $f_x \leqslant 0$ in S_T, $g \in \mathrm{Lip}(\mathbb{R}^2)$ is such that $x \mapsto g(x, y)$ is non-decreasing for every $y \in \mathbb{R}$ and let u be a viscosity solution of* (1.1)–(1.2) *satisfying* (1.3)*. If either*:
- *for every $(y, t) \in \mathbb{R} \times]0, T[$, the function $x \mapsto f(x, y, t)$ is not constant, or*
- *for every $y \in \mathbb{R}$, the function $x \mapsto g(x, y)$ is not constant,*

then $u_x > 0$ in S_T and $u \in C^\infty(S_T)$.

2. Strong solutions

In this section we assume that $f \in \mathrm{Lip}(S_T)$, $g \in \mathrm{Lip}(\mathbb{R}^2)$ and u is the viscosity solution of (1.1) satisfying (1.3). We study some preliminary summability properties of u and its derivatives. Let us recall that $S_T = \mathbb{R}^2 \times]0, T[$ and $u_x = \partial_x u$.

We can always assume that u is a limit of solutions of the regularized problem

$$L_\varepsilon u \equiv \partial_{xx}u + \varepsilon^2\partial_{yy}u + u\partial_y u - \partial_t u = f, \tag{2.1}$$

for $\varepsilon > 0$. Indeed (see [2])

THEOREM 2.1. – *If u is a viscosity solution in the sense of* [13] *of* (1.1)–(1.2) *and* (1.3) *is satisfied, then there exist a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \downarrow 0$ and a sequence (u^{ε_n}) in $C^{2+\alpha,1+\alpha/2}(S_T)\cap \mathrm{Lip}(\overline{S}_T)$ such that for every n the function u^{ε_n} is solution of*

$$L_{\varepsilon_n}u^{\varepsilon_n} = f, \tag{2.2}$$

(u^{ε_n}) converges uniformly on compact subsets of $\mathbb{R}^2\times[0,T[$ to u, as $n\to\infty$ and (1.3) *is satisfied uniformly in ε.*

In order to simplify the exposition, we introduce some notations:

DEFINITION 2.2. – *We define the linear operator*

$$L_u = \partial_{xx} + u\partial_y - \partial_t, \tag{2.3}$$

and, if $b\in L^2_{\mathrm{loc}}(S_T)$, we say that v is a strong solution of

$$L_u v = b, \quad \text{in } S_T, \tag{2.4}$$

if $\partial_{xx}v$, $\partial_y v$, $\partial_t v \in L^2_{\mathrm{loc}}(S_T)$, and equation (2.4) *is satisfied a.e.*

With these notations we prove Proposition 1.2 and also the following result which will be used in the third section:

PROPOSITION 2.3. – *Let u be a viscosity solution of* (1.1) *satisfying* (1.3). *Then u_x is a strong solution of the linear equation $L_u u_x = -u_x u_y + f_x$.*

We first provide some a priori estimates of Caccioppoli type for the derivatives of the functions (u^{ε_n}) and we deduce the proofs of Propositions 1.2 and 2.3 letting n go to infinity.

LEMMA 2.4 (Caccioppoli type inequalities for first derivatives). – *Let u^ε be a C^∞ solution to equation* (2.1) *that satisfies condition* (1.3) *and let $\varphi\in C_0^\infty(S_T)$. There exists a positive constant C_1 which depends only on f,φ and on the constant C_T in* (1.3), *such that*

$$\|u^\varepsilon_{xx}\varphi\|_2 + \|u^\varepsilon_{xy}\varphi\|_2 + \varepsilon\|u^\varepsilon_{yy}\varphi\|_2 + \|u^\varepsilon_t\varphi\|_2 \leqslant C_1,$$

for every positive ε.

Proof. – Let us denote $l = x$ or $l = y$ and let us differentiate equation $L_\varepsilon u^\varepsilon = f$ with respect to the variable l. Then we multiply by $u^\varepsilon_l\varphi^2$ and integrate on S_T. For simplicity, in the remainder of the proof, we omit the index ε in u^ε. Thus we have

$$\int_{S_T}\big(u_{lxx} + \varepsilon^2 u_{lyy} + u_l u_y + u u_{ly} - u_{lt}\big)u_l\varphi^2 = \int_{S_T} f_l u_l\varphi^2.$$

Integrating by parts the first term with respect to x and the second with respect to y, we get

$$\int_{S_T}\big(u_{lx}^2 + \varepsilon^2 u_{ly}^2\big)\varphi^2 = -2\int_{S_T}\big(u_{lx}u_l\varphi\varphi_x + \varepsilon^2 u_{ly}u_l\varphi\varphi_y\big) + I_2 = I_1 + I_2, \tag{2.5}$$

where this equality defines I_1 and

$$I_2 = \int\limits_{S_T} \big(u_l^2 u_y + u u_l u_{ly} - u_l u_{lt} - f_l u_l\big)\varphi^2.$$

By the Cauchy inequality we have

$$I_1 \leqslant \delta \int\limits_{S_T} \big(u_{lx}^2 + \varepsilon^2 u_{ly}^2\big)\varphi^2 + \frac{1}{\delta}\int\limits_{S_T} u_l^2\big(\varphi_x^2 + \varepsilon^2\varphi_y^2\big),$$

for every positive constant δ. In order to estimate I_2 we note that

$$u_l u_{ly} = \partial_y\left(\frac{u_l^2}{2}\right) \quad \text{and} \quad u_l u_{lt} = \partial_t\left(\frac{u_l^2}{2}\right).$$

By parts, we obtain

$$I_2 = \int\limits_{S_T}\left(\frac{1}{2}u_l^2 u_y\varphi^2 - u u_l^2\varphi\varphi_y + u_l^2\varphi\varphi_t\right) - \int\limits_{S_T} f_l u_l \varphi^2.$$

We deduce

$$(1-\delta)\big(\|u_{lx}\varphi\|_2^2 + \varepsilon^2\|u_{ly}\varphi\|_2^2\big) \leqslant \widetilde{C},$$

where $\widetilde{C}$ depends only on C_T in (1.3) and f. Since

$$u_t = u_{xx} + \varepsilon^2 u_{yy} + u u_y - f,$$

we get

$$\|u_t\varphi\|_2 \leqslant \widetilde{C},$$

and the assertion is proved. □

Proof of Proposition 1.2. – It is a simple consequence of Theorem 2.1 and Lemma 2.4, since we can let n go to infinity in (2.2). □

LEMMA 2.5 (Caccioppoli type inequalities for the second derivatives). – *Let u^ε be a solution to equation* (2.1) *that satisfies condition* (1.3) *and let $\varphi \in C_0^\infty(S_T)$. Then there exists a positive constant C_1 that depend only on f, φ and on the constant C_T in* (1.3)*, such that*

$$\big\|u_{xxx}^\varepsilon\varphi\big\|_2 + \varepsilon\big\|u_{xxy}^\varepsilon\varphi\big\|_2 + \varepsilon^2\big\|u_{xyy}^\varepsilon\varphi\big\|_2 + \big\|u_{xt}^\varepsilon\varphi\big\|_2 \leqslant C_1,$$

for every positive ε.

Proof. – We simply outline the proof. We differentiate twice equation $L_\varepsilon u^\varepsilon = f$ with respect to ∂_x, then we multiply by $u_{xx}^\varepsilon\varphi^2$ and integrate on S_T. We find

$$\int\limits_{S_T}\big(u_{xxxx} + \varepsilon^2 u_{xxyy} + \partial_x(u_x u_y + u u_{xy}) - u_{xxt}\big)u_{xx}\varphi^2 = \int\limits_{S_T} f_{xx}u_{xx}\varphi^2.$$

By integrating by parts and arguing as in Lemma 2.4 we readily find the estimate of the first two terms in the statement.

In order to estimate the third term, we differentiate the equation $L_\varepsilon u^\varepsilon = f$ with respect to ∂_x then with respect to ∂_y, we multiply by $\varepsilon^2 u^\varepsilon_{xy}\varphi^2$ and integrate on S_T. Proceeding exactly as before we readily obtain the result. Finally, the estimate of the term u_{xt} follows directly from the identity

$$u_{xt} = u_{xxx} + \varepsilon^2 u_{xyy} + u_x u_y - u u_{xy}.$$

This completes the proof. □

Proof of Proposition 2.3. – Let (u^{ε_n}) a sequence of solutions of the regularized equation, locally uniformly convergent to u. Since every u^{ε_n} belongs to $C^{2+\alpha,1+\alpha/2} \cap W^{3,2}_{\mathrm{loc}}$, the function $v^n = \partial_x u^{\varepsilon_n}$ is a strong solution to the Cauchy problem

$$v_{xx} + \varepsilon_n^2 v_{yy} + v u_y^{\varepsilon_n} + u^{\varepsilon_n} v_y - v_t = f_x, \quad \text{in } S_T, \qquad v = g_x, \quad \text{in } \mathbb{R}^2. \tag{2.6}$$

We can obviously assume by Lemmas 2.4 and 2.5 that (v^n_{xx}), $(\varepsilon_n^2 v^n_{yy})$, $(u^{\varepsilon_n} v^n_y)$ weakly converge to u_{xxx}, 0 and $u u_{xy}$, respectively. Passing to the limit in (2.6), we obtain the thesis. □

3. Classical solutions

In this section we prove that the function u is a classical solution of the equation

$$\partial_{xx} u + (u\partial_y - \partial_t)u = f$$

as defined below. In Section 2 we considered the first-order term

$$(u\partial_y - \partial_t)u \tag{3.1}$$

as a sum of weak derivatives. Here we prove that it is continuous and coincides with the directional derivative w.r.t. the vector $\nu_z = (0, u(z), -1)$:

$$\frac{\partial w}{\partial \nu_z}(z) = \lim_{h\to 0} \frac{w(z + h\nu_z) - w(z)}{h}. \tag{3.2}$$

Then we say that u is a classical solution of $Lu = f$ if the functions u_{xx} and $z \mapsto \frac{\partial u}{\partial \nu_z}(z)$ are continuous and the equation is satisfied at every point of S_T. If w is of class C^1, the derivatives in (3.1) and (3.2) obviously coincide and we will also denote them by $Y_u w$. For less regular functions w, we have:

LEMMA 3.1. – *Le w be a continuous function defined in an open subset Ω of $\mathbb{R}^3$. Assume that its weak derivatives w_y, w_t belong to $L^2_{\mathrm{loc}}(\Omega)$ and that the limit in* (3.2) *exists and is uniform with respect to z in every compact subset of Ω. Then*

$$\frac{\partial w}{\partial \nu_z}(z) = (u\partial_y w - \partial_t w)(z) \quad \textit{a.e. } z \in \Omega.$$

We then denote

$$Y_u w(z) = \frac{\partial w}{\partial \nu_z}(z).$$

Proof. – By Theorem 2.1 there exists a sequence (u^{ε_n}) of smooth functions convergent to u uniformly in Ω and (1.3) is satisfied. Hence we denote

$$\nu_{\varepsilon_n,z} = \bigl(0, -u^{\varepsilon_n}(z), 1\bigr),$$

in order to approximate the directional derivative of w. For every $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \frac{\partial w}{\partial \nu_z}(z)\varphi(z)\,\mathrm{d}z &= \lim_{h\to 0}\int_\Omega \frac{w(z+h\nu_z)-w(z)}{h}\varphi(z)\,\mathrm{d}z\\ &= \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega \frac{w(z+h\nu_{\varepsilon_n,z})-w(z)}{h}\varphi(z)\,\mathrm{d}z.\end{aligned}$$

We can now perform the change of variable $\zeta = \varrho_{\varepsilon_n,h}(z) \equiv z + h\nu_{\varepsilon_n,z}$. Let us estimate the Jacobian determinant independently of ε:

$$\bigl|J_{\varrho^{-1}_{\varepsilon_n,h}}(\zeta)\bigr| = \bigl|1 + hu_y^{\varepsilon_n}\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr)\bigr|^{-1}$$

(since $(u_y^{\varepsilon_n})$ is bounded uniformly with respect to ε and ζ)

$$= 1 - hu_y^{\varepsilon_n}\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr) + hR_{\varepsilon_n,h}(\zeta),$$

where $R_{\varepsilon_n,h}(\zeta) \to 0$ as $h \to 0$ uniformly with respect to ε_n and ζ. Inserting in the previous expression we get:

$$\begin{aligned}\int_\Omega \frac{\partial w}{\partial \nu_z}(z)\varphi(z)\,\mathrm{d}z &= \lim_{h\to 0}\lim_{n\to\infty}\Biggl(\frac{1}{h}\int_\Omega w(\zeta)\varphi\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr)\bigl(1 - hu_y^{\varepsilon_n}\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr) + hR_{\varepsilon_n,h}(\zeta)\bigr)\,\mathrm{d}\zeta\\ &\qquad - \frac{1}{h}\int_\Omega w(\zeta)\varphi(\zeta)h\,\mathrm{d}\zeta\Biggr)\\ &= \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega w(\zeta)\frac{\varphi(\varrho^{-1}_{\varepsilon_n,h}(\zeta)) - \varphi(\zeta)}{h}\,\mathrm{d}\zeta\\ &\qquad - \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega w(\zeta)u_y^{\varepsilon_n}\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr)\varphi\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr)\,\mathrm{d}\zeta\\ &\qquad + \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega w(\zeta)\varphi\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr)R_{\varepsilon_n,h}(\zeta)\,\mathrm{d}\zeta\end{aligned}$$

(using the mean value theorem in the first term, with $\zeta_{\varepsilon_n,h} \in [\zeta, \varrho^{-1}_{\varepsilon_n,h}(\zeta)]$, the change of variable $z = \varrho^{-1}_{\varepsilon_n,h}(\zeta)$ in the second, and the fact that $R_{\varepsilon_n,h} \to 0$ uniformly in the third)

$$\begin{aligned}&= \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega w(\zeta)\bigl\langle \nabla\varphi(\zeta_{\varepsilon_n,h}), \bigl(0, -u^{\varepsilon_n}\bigl(\varrho^{-1}_{\varepsilon_n,h}(\zeta)\bigr), 1\bigr)\bigr\rangle\,\mathrm{d}\zeta\\ &\quad - \lim_{h\to 0}\lim_{n\to\infty}\int_\Omega w\bigl(\varrho_{\varepsilon_n,h}(z)\bigr)u_y^{\varepsilon_n}(z)\varphi(z)\bigl(1 + hu_y^{\varepsilon_n}(z)\bigr)\,\mathrm{d}z.\end{aligned}$$

Due to the uniform boundedness of $\nu_{\varepsilon_n,z}$, we have

$$\varrho^{-1}_{\varepsilon_n,h}(\zeta)\to\zeta,\quad \varrho_{\varepsilon_n,h}(z)\to z,$$

as $h\to 0$ uniformly with respect to ε_n and ζ. Letting h go to 0 we get

$$\int_\Omega \frac{\partial w}{\partial\nu_z}(z)\varphi(z)\,\mathrm{d}z = -\lim_{n\to\infty}\int_\Omega w\big(u^{\varepsilon_n}\varphi_y-\varphi_t\big)-\lim_{n\to\infty}\int_\Omega wu_y^{\varepsilon_n}\varphi$$
$$=-\int_\Omega w\left(u\varphi_y-\varphi_t\right)-\int_\Omega wu_y\varphi,$$

and this completes the proof. □

Let us begin our regularization procedure. The operator L defined in (1.4) is not a Hörmander type operator since it is nonlinear, its coefficients are not smooth and even if we could compute the commutators, we had no information on the structure of the generated Lie algebra. Then, we choose an approximating vector field in such a way that the associated Lie algebra is the simplest non-Abelian one. Fixed a compact set M, for every $z_0\in M$ we define the frozen vector field of order 0 as follows

$$Y_{0,z_0}=\big(u(z_0)+(x-x_0)\big)\partial_y-\partial_t. \tag{3.3}$$

In this way

$$[X,Y_{0,z_0}]=\partial_y$$

and the Lie algebra generated by ∂_x and Y_{0,z_0} spans the whole space at every point. We will call d_{0,z_0} the control distance generated by ∂_x, Y_{0,z_0} and their commutator.

It is known (see for example Remark 2.2 in [12]) that there exist positive constants only dependent on M such that

$$C_1d_{0,z_0}(z_0,z)\leqslant\big(|x-x_0|+|t-t_0|^{1/2}+|y-y_0+u(z_0)(t-t_0)|^{1/3}\big)\leqslant C_2d_{0,z_0}(z_0,z) \tag{3.4}$$

for all $z,z_0\in M$. We call frozen operator of order 0 the operator formally defined as L:

$$L_{0,z_0}=\partial_{xx}+Y_{0,z_0}.$$

This operator has a fundamental solution Γ_{0,z_0} whose asymptotic behaviour can be estimated in terms of the control distance d_{0,z_0} as follows:

$$\big|\Gamma_{0,z_0}(z_0,z)\big|\leqslant Cd_{0,z_0}(z_0,z)^{-Q+2},$$

where $Q=6$ is the so-called homogeneous dimension of the Lie group on $\mathbb{R}^3$ associated to ∂_x, Y_{0,z_0}. We refer to [29,19,27] for more details about this topic.

Using the existence of a fundamental solution it is quite standard to prove:

LEMMA 3.2. – *Let u be a strong solution of* (1.1). *Then u is differentiable with respect to the variable x in S_T. Moreover, for every $\alpha\in\,]0,1[$ and compact subset M of S_T, u_x is Hölder continuous with exponent α in M w.r.t. the distance d_{0,z_0}. Besides there exists $C_3>0$ only dependent on the constant C_T in* (1.3) *such that*

$$\big|u(z)-u(z_0)-u_x(z_0)(x-x_0)\big|\leqslant C_3d^{1+\alpha}_{0,z_0}(z,z_0)\quad\forall z,z_0\in M. \tag{3.5}$$

Proof. – Let us fix a function $\varphi \in C_0^\infty(S_T)$, such that $\varphi = 1$ in M. By definition of fundamental solution, and the fact that u is a strong solution of (1.1), we immediately have

$$(u\varphi)(z) = \int_{S_T} \Gamma_{0,z_0}(z,\zeta)\psi(z_0,\zeta)\,\mathrm{d}\zeta, \quad z \in M,$$

where

$$\psi(z_0,\zeta) = \big(\big(u(z_0) - u(\zeta) + \xi - x_0\big)u_y + f\big)\varphi + uL_{0,z_0}\varphi + 2u_x\varphi_x$$

is a bounded function with compact support. Here we have denoted $\zeta = (\xi,\eta,\tau)$.

Then, we have

$$\partial_x u(z) = \int_{S_T} \partial_x \Gamma_{0,z_0}(z,\zeta)\psi(z_0,\zeta)\,\mathrm{d}\zeta, \quad z \in M,$$

and a standard argument yields the Hölder estimate of u_x. We refer to Theorem 2.16 in [12] for the proof of assertion (3.5). □

Proof of Theorem 1.3. – We have to prove the existence and continuity of the derivatives $\partial_{xx}u$ and $Y_u u$. By brevity, we prove only the second one, which is technically more complicated. In Lemma 3.2, we showed that u can be represented as

$$u(z) = \int_{S_T} \Gamma_{0,z_0}(z,\zeta)(\psi_1 + \psi_2)(z_0,\zeta)\,\mathrm{d}\zeta, \quad z \in M,$$

where

$$\psi_1(z_0,\zeta) = \big(u(z_0) - u(\zeta) + \xi - x_0\big)u_y\varphi, \qquad \psi_2(z_0,\zeta) = f\varphi + uL_{0,z_0}\varphi + 2u_x\varphi_x.$$

It is clear that ψ_2 is Hölder continuous, so that we indicate how to compute the derivative of the term containing ψ_1:

$$\widetilde{u}(z) = \int_{S_T} \Gamma_{0,z_0}(z,\zeta)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta.$$

We denote by χ a $C^\infty([0,+\infty[,[0,1])$ function such that

$$\chi(s) = 0, \quad \text{for } s \leqslant \frac{1}{2}, \qquad \chi(s) = 1, \quad \text{for } s \geqslant 1,$$

and we define

$$u_\delta(z) = \int_{S_T} \Gamma_{0,z_0}(z,\zeta)\chi\left(\frac{d_{0,z_0}(z,\zeta)}{\delta}\right)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta,$$

and

$$v(z_0) = \int_{S_T} Y_u\Gamma_{0,z_0}(z_0,\zeta)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta.$$

Using the local behavior of $\Gamma_{0,z_0}(z,\zeta)$ and the fact that $|\psi_1(z_0,\zeta)| \leqslant C d_{0,z_0}(z_0,\zeta)$ we get, for every z, z_0 such that $d_{0,z_0}(z_0,z) \leqslant \delta$

$$\left|u_\delta(z) - \widetilde{u}(z)\right| \leqslant C \int\limits_{d_{0,z_0}(z,\zeta)\leqslant\delta} d_{0,z_0}^{-Q+2}(z,\zeta) d_{0,z_0}(z_0,\zeta)\,\mathrm{d}\zeta$$

(using the fact that $d_{0,z_0}(z_0,\zeta) \leqslant C_0(d_{0,z_0}(z,\zeta) + d_{0,z_0}(z_0,z)) \leqslant 2C_0\delta$ and the polar coordinates associated to the homogeneous Lie group)

$$\leqslant C\delta \int\limits_0^\delta \rho^{-Q+2+Q-1}\,\mathrm{d}\rho = C\delta^3$$

and analogously

$$\left|Y_u u_\delta(z_0) - v(z_0)\right| \leqslant \delta, \qquad \sup_{d_{0,z_0}(z_0,z)\leqslant\delta} \left|\partial_y u_\delta(z)\right| \leqslant C\log(\delta).$$

Then the derivative $\frac{\partial u}{\partial \nu_{z_0}}(z_0)$ can now be computed as follows:

$$\begin{aligned}
&\left|\frac{\widetilde{u}(z_0+\delta\nu_{z_0}) - \widetilde{u}(z_0)}{\delta} - v(z_0)\right| \\
&\quad = \frac{|\widetilde{u}(z_0+\delta\nu_{z_0}) - u_\delta(z_0+\delta\nu_{z_0})|}{\delta} + \left|\frac{u_\delta(z_0+\delta\nu_{z_0}) - u_\delta(z_0)}{\delta} - v(z_0)\right| \\
&\qquad + \frac{|\widetilde{u}(z_0) - u_\delta(z_0)|}{\delta}
\end{aligned} \tag{3.6}$$

(by mean value theorem, for some $\widetilde{\delta} \in [0,\delta]$)

$$\begin{aligned}
&\leqslant \delta^2 + \left|u(z_0)\partial_y u_\delta(z_0+\widetilde{\delta}\nu_{z_0}) - \partial_t u_\delta(z_0+\widetilde{\delta}\nu_{z_0}) - v(z_0)\right| \\
&= \left|u(z_0) - u(z_0+\widetilde{\delta}\nu_{z_0})\right|\left|\partial_y u_\delta(z_0+\widetilde{\delta}\nu_{z_0})\right| + \left|Y_u u_\delta(z_0+\widetilde{\delta}\nu_{z_0}) - v(z_0)\right| \\
&\leqslant \widetilde{\delta}^{1/2}\log(\widetilde{\delta}) + \widetilde{\delta} + \delta \to 0, \quad \text{as } \delta \to 0.
\end{aligned}$$

Then

$$v(z_0) = \frac{\partial\widetilde{u}}{\partial\nu_{z_0}}(z_0),$$

and v is continuous, since this limit is uniform. Finally, by Lemma 3.1,

$$Y_u u(z_0) = v(z_0) + \int\limits_{S_T} Y_u \Gamma_{0,z_0}(z_0,\zeta)\psi_2(z_0,\zeta)\,\mathrm{d}\zeta,$$

and it is a continuous function. □

Property (3.5) allows us to introduce new vector fields frozen of order 1:

$$X = \partial_x, \qquad Y_{1,z_0} = \left(u(z_0) + u_x(z_0)(x-x_0) + \frac{(x-x_0)^2}{2}\right)\partial_y - \partial_t. \tag{3.7}$$

Note that, even if we have no information on u_x, the rank of the Lie algebra generated by ∂_x and Y_{1,z_0} is constant at every point. Indeed if $u_x(z_0) \neq 0$, then

$$\partial_x, \quad Y_{1,z_0}, \quad [\partial_x, Y_{1,z_0}] = \big(u_x(z_0) + 2(x - x_0)\big)\partial_y$$

are linearly independent near z_0. On the other hand if $u_x(z_0) = 0$ then $[\partial_x, Y_{1,z_0}](z_0) = 0$ and we need commutators of order 3 to span the whole space:

$$\partial_x, \quad Y_{1,z_0}, \quad \big[\partial_x, [\partial_x, Y_{1,z_0}]\big] = \partial_y.$$

The intrinsic gradient is the vector

$$\nabla_{1,z_0} = \big(\partial_x, Y_{1,z_0}, [\partial_x, Y_{1,z_0}], \big[\partial_x, [\partial_x, Y_{1,z_0}]\big]\big)$$

and $(\nabla_{1,z_0})_i$ will denote its components. The associated control distance can be defined as follows:

DEFINITION 3.3. – *For every $z, z_0, \bar{z}$ there exist constants $\theta_1, \theta_2, \theta_4$, such that*

$$z = \exp\big(\theta_1(\nabla_{1,z_0})_1 + \theta_2(\nabla_{1,z_0})_2 + \theta_4(\nabla_{1,z_0})_4\big)(\bar{z}).$$

Precisely

$$\theta_1 = x - \bar{x}, \qquad \theta_2 = -(t - \bar{t}),$$

$$\theta_4 = y - \bar{y} + (t - \bar{t})\Bigg(u(z_0) + u_x(z_0)\bigg(\frac{x - \bar{x}}{2} + \bar{x} - x_0\bigg) + \frac{(x - \bar{x})^2}{2} + \frac{(x - \bar{x})(\bar{x} - x_0)}{2} + \frac{(x_0 - \bar{x})^2}{6}\Bigg).$$

If $2u_x(z_0) + x - \bar{x} + 2(\bar{x} - x_0) \neq 0$, then there also exists a constant θ_3 such that

$$z = \exp\big(\theta_1(\nabla_{1,z_0})_1 + \theta_2(\nabla_{1,z_0})_2 + \theta_3(\nabla_{1,z_0})_3\big)(\bar{z})$$

and

$$\theta_3 = \frac{2\theta_4}{2u_x(z_0) + (x - \bar{x}) + 2(\bar{x} - x_0)}.$$

Then the control distance associated to d_{1,z_0} can be defined as

$$d_{1,z_0}(z, \bar{z}) = |\theta_1| + |\theta_2|^{1/2} + \min\big\{|\theta_3|^{1/3}, |\theta_4|^{1/4}\big\}.$$

The function d_{1,z_0} has been introduced in [27], where it is also proved that it is a quasi-distance locally equivalent to the Carnot–Caratheodory's one.

Let us note explicitly that d_{1,z_0} is not an homogeneous function, so that the group associated to this choice of vector fields is not homogeneous. However the metric is doubling. In other words there exists a constant $C > 0$ only dependent on the fixed compact set M such that

$$\big|B_{1,z_0}(z, 2R)\big| \leqslant C\big|B_{1,z_0}(z, R)\big|,$$

for every $z \in M$, where $B_{1,z_0}(z, 2R)$ denotes the ball of the metric d_{1,z_0}, and $|\cdot|$ the Lebesgue measure. Let us also note that the distances d_{0,z_0} and d_{1,z_0} are not equivalent, and the following relation holds:

Remark 3.4. –

$$d_{0,z_0}(z, z_0) \leqslant C d_{1,z_0}(z, z_0).$$

Proof. – If $\min\{|\theta_3|^{1/3}, |\theta_4|^{1/4}\} = |\theta_3|^{1/3}$ then

$$d_{0,z_0}(z, z_0) \leqslant |\theta_1| + |\theta_2|^{1/2} + \left| y - y_0 + (t - t_0)\left(u(z_0) + \frac{u_x(z_0)(x - x_0)}{2} \right) \right|^{1/3}$$

(since u_x is bounded)

$$\leqslant C\big(|\theta_1| + |\theta_2|^{1/2} + |\theta_3|^{1/3} + |t - t_0|^{1/3}|x - x_0|^{2/3}\big) \leqslant d_{1,z_0}(z_0, z).$$

Analogously, if $\min\{|\theta_3|^{1/3}, |\theta_4|^{1/4}\} = |\theta_4|^{1/4}$

$$d_{0,z_0}(z, z_0) \leqslant |\theta_1| + |\theta_2|^{1/2} + |\theta_4|^{1/4} + |t - t_0|^{1/4}|x - x_0|^{1/2} \leqslant d_{1,z_0}(z_0, z).$$

From Lemma 3.2 and the preceding remark it follows that

COROLLARY 3.5. – *If u is a Lipschitz continuous, strong solution of* (1.1)*, then for every $\alpha \in]0, 1[$ for every compact set M there exists a constant $C_3 > 0$ and only dependent on C_T in* (1.3)*, such that*

$$\big| u(z) - u(z_0) - u_x(z_0)(x - x_0) \big| \leqslant C_3 d_{1,z_0}^{1+\alpha}(z, z_0).$$

In order to study the regularity of the solution, we proceed as before, using the fundamental solution of a suitable operator defined in terms of Y_{1,z_0}. We call frozen operator of order 1:

$$L_{1,z_0} = \partial_{xx} + Y_{1,z_0}$$

and Γ_{1,z_0} will be its fundamental solution. It satisfies the following estimate

$$\Gamma_{1,z_0}(z, \zeta) \leqslant C \frac{d_{1,z_0}^2(z, \zeta)}{|B_{1,z_0}(z, d_{1,z_0}(z, \zeta))|}. \tag{3.8}$$

PROPOSITION 3.6. – *If u is a strong solution of* (1.1) *satisfying* (1.3) *then the weak derivative $Y_u u_x$ is continuous.*

Proof. – Arguing as in Lemma 3.2, we see that u can be represented in terms of the fundamental solution Γ_{1,z_0} and that

$$\partial_x u(z) = \int_{S_T} \partial_x \Gamma_{1,z_0}(z, \zeta)(\psi_1 + \psi_2)(z_0, \zeta)\, \mathrm{d}\zeta, \quad z \in M,$$

where

$$\psi_1(z_0, \zeta) = -\left(u(\zeta) - u(z_0) - u_x(z_0)(x - x_0) - \frac{(x - x_0)^2}{2} \right) u_y \varphi,$$

$$\psi_2(z_0, z) = f\varphi + u L_{1,z_0}\varphi + 2u_x\varphi_x.$$

As before we indicate how to compute the derivative of the term containing ψ_1:

$$\partial_x \widetilde{u}(z) = \int_{S_T} \partial_x \Gamma_{1,z_0}(z,\zeta)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta.$$

We denote by χ the same function as in Lemma 3.2 and we define

$$\widetilde{u}_\delta(z) = \int_{S_T} \partial_x \Gamma_{1,z_0}(z,\zeta)\chi\left(\frac{d_{1,z_0}(z,\zeta)}{\delta}\right)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta$$

and

$$v(z_0) = \int_{S_T} Y_u \partial_x \Gamma_{1,z_0}(z_0,\zeta)\psi_1(z_0,\zeta)\,\mathrm{d}\zeta.$$

We remark explicitly that the last integral is convergent by Corollary 3.5 and estimate (3.8). In order to proceed as in the proof of Theorem 1.3, we need to prove that

$$\left|\widetilde{u}_\delta(z) - \widetilde{u}_x(z)\right| \leqslant \delta^{2+\alpha}, \qquad \left|Y_u \widetilde{u}_\delta(z_0) - v(z_0)\right| \leqslant \delta^{\alpha} \tag{3.9}$$

for z, z_0 satisfying $d_{1,z_0}(z_0,z) \leqslant \delta$. However we can not repeat the same proof, since no polar coordinates are associated to the Lie algebra generated by ∂_x, Y_{1,z_0}, which is not homogeneous. We use instead the doubling property of the metric d_{1,z_0}:

$$\begin{aligned}
\left|\widetilde{u}_\delta(z) - \widetilde{u}_x(z)\right| &= \int_{d_{1,z_0}(z,\zeta)\leqslant\delta} \left|\partial_x \Gamma_{1,z_0}(z,\zeta)\psi_1(z_0,\zeta)\right| \mathrm{d}\zeta \\
&\leqslant C \int_{d_{1,z_0}(z,\zeta)\leqslant\delta} \frac{d_{1,z_0}(z,\zeta)}{|B_{1,z_0}(z, d_{1,z_0}(z,\zeta))|}\left(d_{1,z_0}^{1+\alpha}(z,\zeta) + d_{1,z_0}^{1+\alpha}(z,z_0)\right)\mathrm{d}\zeta \\
&\leqslant C \sum_{k=0}^{\infty} \int_{\frac{\delta}{2^{k+1}}\leqslant d_{1,z_0}(z,\zeta)\leqslant\frac{\delta}{2^k}} \frac{d_{1,z_0}^{2+\alpha}(z,\zeta)}{|B_{1,z_0}(z, d_{1,z_0}(z,\zeta))|}\,\mathrm{d}\zeta \\
&\quad + C\delta^{1+\alpha} \sum_{k=0}^{\infty} \int_{\frac{\delta}{2^{k+1}}\leqslant d_{1,z_0}(z,\zeta)\leqslant\frac{\delta}{2^k}} \frac{d_{1,z_0}(z,\zeta)}{|B_{1,z_0}(z, d_{1,z_0}(z,\zeta))|}\,\mathrm{d}\zeta \\
&\quad + C\sum_{k=0}^{\infty}\left(\frac{\delta}{2^k}\right)^{2+\alpha} \frac{|B_{1,z_0}(z,\frac{\delta}{2^{k+1}})|}{|B_{1,z_0}(z,\frac{\delta}{2^k})|} + C\delta^{1+\alpha}\sum_{k=0}^{\infty}\frac{\delta}{2^k}\frac{|B_{1,z_0}(z,\frac{\delta}{2^{k+1}})|}{|B_{1,z_0}(z,\frac{\delta}{2^k})|}
\end{aligned}$$

(since the metric is doubling)

$$\leqslant C\sum_{k=0}^{\infty}\left(\frac{\delta}{2^k}\right)^{2+\alpha} + C\delta^{1+\alpha}\sum_{k=0}^{\infty}\frac{\delta}{2^k} \leqslant C\delta^{2+\alpha}.$$

The proof of the second assertion in (3.9) is analogous. Once these properties are established, arguing as in (3.6), we deduce that

$$Y_u \widetilde{u}_x(z_0) = v(z_0)$$

and

$$Y_u u_x(z_0) = -\int_{S_T} Y_u \Gamma_{1,z_0}(z_0,\zeta)\partial_x \psi_2(z_0,\zeta)\,\mathrm{d}\zeta + v(z_0). \qquad \square$$

4. Propagation principle and smoothness

In this section we consider the classical solution u of the Cauchy problem (1.1)–(1.2) satisfying (1.3) and we prove Theorem 1.4. We aim to show that a propagation principle for minima of u_x holds, so that

$$u_x(z) > 0, \quad \text{in } S_T,$$

then Theorem 1.4 follows from Theorem 1.1.

We recall some classical results due to Bony [3] about the propagation of maxima. Let Ω be an open connected subset of $\mathbb{R}^N$ and

$$D : \Omega \to \mathbb{R}^N$$

a locally Lipschitz continuous vector field. A non-empty subset E of Ω, relatively closed in Ω, is said to be positively D-invariant if for every curve

$$\gamma : [0, S] \to \Omega$$

such that $\gamma' = D(\gamma)$ and $\gamma(0) \in E$, we necessarily have $\gamma(s) \in E$ for every $s \in [0, S]$. If E is positively invariant for D and $-D$, we say that E is D-invariant. In other words, E is D-invariant if for every

$$\gamma : I \to \Omega$$

integral curve of D such that $\gamma(s_0) \in E$ for some $s_0 \in I$, then $\gamma(I) \subseteq E$.

Positive D-invariant sets can be characterized in a remarkable geometric way. A vector $\nu \in \mathbb{R}^N \setminus \{0\}$ is said to be an exterior normal to E in $z \in E$ (in symbols, $\nu \perp E$ in z) if

$$B\big(z + \nu, |\nu|\big) \cap E = \emptyset,$$

where B is the Euclidean ball

$$B(z, R) = \big\{\zeta \in \mathbb{R}^N \mid |z - \zeta| < R\big\}.$$

We put:

$$E^* = \big\{z \in E \mid \exists \nu \perp E \text{ in } z\big\}.$$

It is easy to show that $E^* \neq \emptyset$ whenever $\emptyset \neq E \neq \Omega$.

THEOREM 4.1 (Bony). – *Let $E \subseteq \Omega$, relatively closed in Ω. Then E is positive D-invariant if and only if*

$$\big\langle D(z), \nu\big\rangle \leqslant 0, \quad \forall z \in E^*,\ \forall \nu \perp E \textit{ in } z.$$

In particular, E is D-invariant if and only if

$$\big\langle D(z), \nu\big\rangle = 0, \quad \forall z \in E^*,\ \forall \nu \perp E \textit{ in } z.$$

In order to apply the theorem, we prove a Hopf type lemma for the set

$$E = \{ z \in S_T \mid u_x(z) = 0 \}, \tag{4.1}$$

by using some functions introduced in [26], Proposition 6.1, for the study of a boundary value problem for operators related to the linear operator L_u in Definition 2.2.

LEMMA 4.2. – *Let E be as in* (4.1). *For every $z_0 \in E^*$ and $\nu = (\nu_x, \nu_y, \nu_t) \perp E$ in z_0, we have*

$$\langle X, \nu \rangle = \nu_x = 0 \quad \textit{and} \quad \langle Y_u(z_0), \nu \rangle = u(z_0)\nu_y - \nu_t \leqslant 0.$$

Proof. – We define $\widetilde{L}$ by $\widetilde{L}w = L_u w + u_y w$. By Proposition 2.3, $\widetilde{L}u_x = f_x$ in S_T and, by the maximum principle and our assumption on f and g, we find $u_x \geqslant 0$ in S_T. Hence $z_0 \in E^*$ is a minimum point for u_x. Since, by Theorem 1.3 and Proposition 3.6, u_{xx} and Yu_x are defined and continuous, we have

$$u_{xx}(z_0) = 0 \quad \text{and} \quad Yu_x(z_0) = 0. \tag{4.2}$$

To prove the first assertion, we suppose, by contradiction, that $\langle X, \nu \rangle \neq 0$. We set $\bar{z} = z_0 + \nu$, $r = |\nu|$ and

$$w(z) = \mathrm{e}^{-\lambda |z - \bar{z}|^2} - \mathrm{e}^{-\lambda r^2},$$

for some positive λ. A straightforward computation yields

$$\widetilde{L}w(z_0) = 2\lambda \mathrm{e}^{-\lambda |\nu|^2} \left(2\lambda \nu_x^2 - 1 + u(z_0)\nu_y - \nu_t\right).$$

Thus, there exist $\lambda, \varrho > 0$ such that

$$\widetilde{L}w(z) > 0, \quad \forall z \in \Omega_0 \equiv B(\bar{z}, r) \cap B(z_0, \varrho).$$

Since $u_x > 0$ in $B(\bar{z}, r) \cap \partial B(z_0, \varrho)$, $u_x \geqslant 0$ in $\partial B(\bar{z}, r)$, and $w = 0$ in $\partial B(\bar{z}, r)$, there exists a positive δ such that $u_x - \delta w \geqslant 0$ in $\partial \Omega_0$ and $\widetilde{L}(u_x - \delta w) < 0$ in Ω_0, by the minimum principle we get

$$u_x \geqslant \delta w, \quad \text{in } \Omega_0. \tag{4.3}$$

As noticed above, $u_{xx}(z_0)$ exists and, we get from (4.3) that

$$\begin{aligned} u_{xx}(z_0) &= \lim_{h \to 0^+} \frac{u_x(x_0 + h\nu_x, y_0, t_0) - u_x(z_0)}{h} \\ &\geqslant \lim_{h \to 0^+} \delta \frac{w(x_0 + h\nu_x, y_0, t_0) - w(z_0)}{h} = 2\delta\lambda \nu_x^2 \mathrm{e}^{-\lambda r^2} > 0. \end{aligned}$$

This inequality contradicts (4.2) and proves the first claim. As a consequence, by Theorem 4.1, E is X-invariant, that is

$$z_0 = (x_0, y_0, t_0) \in E \quad \Longrightarrow \quad \{ (x, y_0, t_0) \mid x \in \mathbb{R} \} \subseteq E.$$

Therefore, for every $z_0 \in E^*$ and $\nu \perp E$ in z_0, we have

$$\{ (x, y, t) \in \mathbb{R}^3 \mid (y - y_0 - \nu_y)^2 + (t - t_0 - \nu_t)^2 < |\nu|^2 \} \cap E = \emptyset. \tag{4.4}$$

In order to prove the second claim we suppose, by contradiction, that $u(z_0)\nu_y - \nu_t > 0$. For a positive r we denote $\bar{z} = z_0 + r\nu$ and

$$\widetilde{B}(\bar{z}, r) = \left\{(x, y, t) \in \mathbb{R}^3 \mid r^2(x-\bar{x})^2 + (y-\bar{y})^2 + (t-\bar{t})^2 < r^2|\nu|^2\right\}.$$

By (4.4), $\widetilde{B}(\bar{z}, r) \cap E = \emptyset$ for every $r \in]0, 1]$. If we choose $r < u(z_0)\nu_y - \nu_t$ and let:

$$v(x, y, t) = \mathrm{e}^{-r^2(x-\bar{x})^2-(y-\bar{y})^2-(t-\bar{t})^2} - \mathrm{e}^{-r^2|\nu|^2},$$

a direct computation shows that

$$\widetilde{L}v(z_0) = 2r\mathrm{e}^{-r^2|\nu|^2}\big(u(z_0)\nu_y - \nu_t - r\big) > 0,$$

then there exists $\varrho > 0$ such that

$$\widetilde{L}v(z) > 0, \quad \forall z \in \widetilde{\Omega}_0 \equiv \widetilde{B}(\bar{z}, r) \cap B(z_0, \varrho).$$

As in the previous case, it is easy to see that there exists $\delta > 0$ such that

$$u_x \geqslant \delta v, \quad \text{in } \widetilde{\Omega}_0$$

so that

$$Y_u u_x(z_0) \geqslant \delta Y_u v(z_0) = 2\delta r\mathrm{e}^{-r^2|\nu|^2}\big(u(z_0)\nu_y - \nu_t\big) > 0.$$

This inequality contradicts (4.2) and completes the proof. □

Proof of Theorem 1.4. – As stated above, we show that condition (1.7) is satisfied by proving that the set E defined in (4.1) is empty.

Suppose, by contradiction, that there exists $z_0 = (x_0, y_0, t_0) \in E$. By Lemma 4.2, E is X-invariant, then it follows from (4.2) that $f_x(x, y_0, t_0) = Yu_x(x, y_0, t_0) = 0$ for every $x \in \mathbb{R}$ and this contradicts our assumption on f.

In the other case, we observe that, by (1.3), the integral curve of Y_u starting at z_0

$$\gamma(s) = \left(x_0, y_0 + \int_0^s u(\gamma(\tau))\,\mathrm{d}\tau, t_0 - s\right)$$

is defined for every $s \in [0, t_0]$. By Lemma 4.2 and Theorem 4.1, E is positively Y_u-invariant. As a consequence $\gamma([0, t_0[) \subseteq E$ and using again the X-invariance of E and the continuity of u we find

$$u\left(x, y_0 + \int_0^{t_0} u\big(\gamma(\tau)\big)\,\mathrm{d}\tau, 0\right) = u\left(x_0, y_0 + \int_0^{t_0} u\big(\gamma(\tau)\big)\,\mathrm{d}\tau, 0\right)$$

for any $x \in \mathbb{R}$ and this contradicts our assumption on g. In both cases we have $u_x > 0$ in S_T and the thesis follows from Theorem 1.1. □

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